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Approximation of the Sum of a Power Series by Its First Four Terms

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Abstract: The paper develops a three-parameter method for approximating the sum of the McLaurin series by its first four expansion terms, which allows obtaining analytical approximations for functions that are expanded into a power series. The expressions for the approximation parameters (a, b, c) of the exact sum ∑(S) of the geometric power series-base are obtained in general form and are determined by the coefficients at the second (A), third (B), and fourth (C) terms of the McLaurin series. For series that converge rapidly {their coefficients satisfy the inequality $(a_n)^2 \geq (a_{n-1} \times a_{n+1})$}, the new method gives the real values of the sum ∑(S), and for series that converge slowly {for them $(a_n)^2 < (a_{n-1} \times a_{n+1})$}, the method gives the complex-conjugate roots of the parameters of their sum ∑(S). The paper presents examples of approximate determination of series sums by both three-parameter and two-parameter methods based on the analysis of series coefficients. The accuracy of the two- and three-parameter methods of approximation of ∑(S) is evaluated on the basis of determining the approximate sums of known numerical series (for the number π, the number e, etc.). The new three-parameter method was used to approximate the sum of a series whose first terms were obtained by Lord Rayleigh when refining the method of determining the capillary complex of a liquid by the capillary rise method.

Keywords: Sum of the Series Approximation, Three-Parameter Approximation, McLaurin's Sum Series Approximation, Sum of Numerical Series Estimation, Rayleigh's Series Decomposition, Rayleigh's Sum Series Calculation

1. Introduction

In the theoretical analysis of fundamentally important regularities that constitute our understanding of a new phenomenon, the role of analytical methods remains extremely high. Cole [1] notes that their role is also high in the qualitative determination of the parameters of physical phenomena. Therefore, he points out the importance of various methods of perturbation theory, which are the main analytical tool for studying nonlinear physical and engineering problems. In reality, only a few terms of the perturbed expansion can be calculated, usually no more than two or three. The resulting series often converge slowly or even diverge. Nevertheless, these few terms contain significant information from which it is necessary to extract the maximum possible, summarizes Van Dyke [2].

However, after the appearance of the works of Cantor et al. on set theory, mathematics was formed into a single whole, i.e., a complete science with its own subject and method. And now "refined" mathematicians no longer consider the purpose of mathematics to be the development of the "language" of natural science, i.e., the apparatus for solving problems in the exact sciences [3]. Modern mathematicians deal with the problems of the structure of mathematics itself and its individual aspects. The problems of a "language" for the natural sciences are now dealt with only by specialists in applied mathematics. It is this weakening of the interest of pure mathematicians in the problems of approximating the sum of series that explains why unresolved questions and some unsolved problems can still be found in this area of higher mathematics.

2. Literature Review

Historically, the first method to accelerate the convergence of power series is the fractional rational transformation of a variable in the form of the Euler transform: $x = \varepsilon/(1+\varepsilon)$, [2, 3]. The purpose of the Euler transform is to transfer the feature ε = −1 to an infinitely distant point. In this case, if there are no other features in the complex plane, the radius of convergence becomes infinite.

In general, the problem of approximate determination of the sum of an infinite series is solved by approximation methods ∑(S) by the first few coefficients of the power series expansion.

Among the low-parameter methods for approximating the sum of a power series in mechanics, the Shanks method [2] is well known. It consists in approximating the sum of the McLaurin series by its first three terms. Applying the nonlinear Shanks transformation to the first three terms of the power series $S(\varepsilon) = 1+a\varepsilon+b\varepsilon^2\ldots$ gives a simple rational fraction:

$$\sum(S) \cong \frac{a+(a^2-b)\cdot\varepsilon}{a-b\cdot\varepsilon}, \quad (1)$$

which is often a more accurate approximation to the sum of the series than its fragment, which is the sum of the first three terms. For example, this sum gives the exact value if the series is a geometric progression (either convergent or divergent). A sequence of series terms is called a geometric progression because there is a relationship between adjacent terms: that is, each term is the geometric mean of the terms preceding it and the term following it. By the way, in the fundamental Handbook [4, 5], in the Section of Functional Power Series, which has an exact sum, under No. 1, there is a series S, which is a geometric progression whose sum is $\sum x^k = (1-x)^{-1}$, provided that $x\neq1$. Thus, we can say that power series of the form $\sum p(k)\cdot x^k$ – generalize geometric series that have an analytical expression for their exact sum. This sum ∑(S) is finite for $x \neq 1$ [4, 5], and the series $S(x)$ diverges at the point $x \to 1$ ($\sum S(x)\to\infty$).

Other series of geometric type, i.e., those that have an analytical expression for their exact sum, can be cited as examples:

a series $S(x) = x+2x^2+3x^3+4x^4+\ldots$ has a sum $\sum(S) = \dfrac{x}{(1-x)^2}$, (2)

a series $S(x) = 1+3x+5x^2+7x^3+\ldots$ has a sum $\sum(S) = \dfrac{1+x}{(1-x)^2}$. (3)

The most powerful method of approximation is the Pade method [5], in which it is carried out within the framework of approximating the sum of a series by rational functions. However, for many cases, the Pade approximation method is not acceptable in principle. For example, when only a few expansion terms are obtained in the solution by the small parameter method, the coefficients of which must have a certain physical interpretation.

In this sense, the most promising method for approximating the sum of a power series by its first three terms was the method of Ludanov, whose abstract (9B 832 DEP) was presented in the journal Mathematics about forty years ago (in No. 9 of 1984). In this article, a method of two-parameter approximation of the McLaurin power series by its first three terms was developed. A comparison [6, 7] showed that this method was much more accurate than the Pade method (ceteris paribus).

3. Formulation of the Problem

In cases where the coincidence of only the first two derivatives of a formula that approximately describes an exact expression or a fragment of a power series in which this expression is expanded is sufficient, the optimal solution to the problem is the method of two-parameter approximation of the sum of the McLaurin series given in the author's article [8].

The authors developed a method of two-parameter approximation of the McLaurin power series by its first three terms in the form of the Nth degree of an arbitrary analytic function $y(x)$, in which the variable x is written as the product of the new variable ε and the parameter M ($x = M\cdot\varepsilon$). Such a substitution makes it possible to "stretch" or "compress" the variable x and get another parameter when constructing a multi-parameter approximation expression. The point is that each additional approximation parameter gives a sharp increase in the accuracy of the approximation. For example, the error of a two-parameter approximation [8] is estimated as $R(x^3)$, while for a three-parameter approximation, it is obvious that the error will be an order of magnitude lower than $R(x^4)$.

The most promising option for developing a three-parameter approximation is the geometric-type McLaurin power series $S(x)$, presented in Dwight's Handbook [9]:

$$S(x) = 1+\boldsymbol{a}\cdot x+(\boldsymbol{a}+\boldsymbol{b})\cdot x^2+(\boldsymbol{a}+2\boldsymbol{b})\cdot x^3+\ldots \quad (4)$$

It also has an analytical expression for the exact sum:

$$\sum(S) = 1+\frac{\boldsymbol{a}\cdot x+(\boldsymbol{b}-\boldsymbol{a})\cdot x^2}{(1-x)^2} \text{ provided that } x\neq 1. \quad (5)$$

This power series S and the generalized expression of its sum ∑(S) already include two parameters (a and b), and if we try to include the third parameter c in the expression S and ∑(S), we can obtain a generalized expression for the approximant of the sum of the McLaurin series by its first four terms:

$$S(x) = 1+A\cdot x+B\cdot x^2+C\cdot x^3+\ \ldots, \text{ where } \sum S(x) \cong f(x, A, B, C). \quad (6)$$

4. Research Results

4.1. Approximation of the Sum of the Mclaurin Series by the First Four Terms

Here we will use a technique that was used in the author's previous work [8]. If the variable x includes the multiplier *c*, we get the "compressed" or "stretched" variable ε in the form $x=\varepsilon/c$. Substituting the new variable ε/c instead of *x* (this is convenient for obtaining the same dimensionality), we obtain the expression for the three-parameter approximation:

$$S(x) \to S\left(\frac{\varepsilon}{c}\right) = 1 + \boldsymbol{a}\cdot\left(\frac{\varepsilon}{c}\right) + (\boldsymbol{a}+\boldsymbol{b})\cdot\left(\frac{\varepsilon}{c}\right)^2 + (\boldsymbol{a}+2\boldsymbol{b})\cdot\left(\frac{\varepsilon}{c}\right)^3 + \ldots \tag{7}$$

Thus, here we have a modified power series S(ε), which has a different expression for its exact sum ∑(S):

$$S(\varepsilon)=1+\frac{\boldsymbol{a}}{\boldsymbol{c}}\cdot\varepsilon+\frac{\boldsymbol{a}+\boldsymbol{b}}{\boldsymbol{c}^2}\cdot\varepsilon^2+\frac{\boldsymbol{a}+2\boldsymbol{b}}{\boldsymbol{c}^3}\cdot\varepsilon^3+\ldots \tag{8}$$

Let's introduce the notations: a/c = A, $(a+b)/c^2$ = B, and $(a+2b)/c^3$ = C, substitute them into the exact sum of the series ∑(S), and then multiply the numerator and denominator of the fraction by c^2 and after the reductions we get the modified series and the estimate of its sum:

$$S(\varepsilon) = 1 + A\cdot\varepsilon + B\cdot\varepsilon^2 + C\cdot\varepsilon^3 + \ldots, \tag{9}$$

$$\sum S(\varepsilon) \cong 1 + \frac{(\boldsymbol{a}\cdot\boldsymbol{c})\cdot\varepsilon + (\boldsymbol{b}-\boldsymbol{a})\cdot\varepsilon^2}{(\boldsymbol{c}-\varepsilon)^2}. \tag{10}$$

We also get a system of three equations with three unknowns *a*, *b*, and *c*:

$$\begin{cases} A = \boldsymbol{a}/\boldsymbol{c} \\ B = (\boldsymbol{a}+\boldsymbol{b})/\boldsymbol{c}^2 \\ C = (\boldsymbol{a}+2\boldsymbol{b})/\boldsymbol{c}^3 \end{cases} \tag{11}$$

By solving the resulting system of equations, we can express the coefficients of the approximated series A, B, and C through lowercase letters - the parameters of the sum of the series *a*, *b* and *c*. The definition of expressions for *a*, *b* and *c* through the coefficients A, B, and C is as follows. Since the small *b* is included in only two equations of the system out of three, by transforming the second and third equations of the system, we find *b* from them, and by equating them - we exclude it from the system. Then, from the first designation of the coefficient A, we find c and, substituting the value of *a*/A instead, we obtain a quadratic equation for *a*, solving which we find the roots:

$$\boldsymbol{a}_{1,2} = \left(\frac{A}{C}\right)\cdot[B \pm \sqrt{(B^2 - A\cdot C)}], \text{ where Det=}B^2\text{-A·C}. \tag{12}$$

Based on the found parameter *a*, it is easy to find the others from the above notation: $c = A/a$, and $b = B\cdot c^2 - a$. If the root expressions of $a_{1,2}$ have a negative root expression (Det=B^2-A·C) Det<0, then there will be complex-conjugate roots, and the expression of the sum of the series ∑(S) will also be complex - but these solutions are not considered in this paper.

4.2. Approximating the Sum of Number Series to Calculate the Number π

More than six centuries ago, mathematicians invented series and began to use them to calculate mathematical constants, including the number π. To test the accuracy of the approximation of these series using two- and three-parameter methods, let's try to determine their sums and estimate the error.

Example 1.

As a first example, let's consider probably the most famous series in mathematics - the numerical series of Ramadge of Sanhavara, obtained by him in early 1400 AD [10]. It is a series of inverse values of odd numbers of a normal numerical sequence, the sum of which was first used by Ramaja to accurately calculate the number π:

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots, \text{ where } \sum(S) = \frac{\pi}{4}. \tag{13}$$

Let's transform this number series into the McLaurin series:

$$S = 1 - \frac{1}{3}x + \frac{1}{5}x^2 - \frac{1}{7}x^3 + \frac{1}{9}x^4 + \ldots, \text{ where } x=1. \tag{14}$$

Let's estimate the value of π by approximating the sum of this series using different methods: the two-parameter methods of Shanks (for him, a = -1/3; b = +1/5), and the author (MA No. 1 - using Newton's binomial) and the three-parameter approximation method (MA No. 2) presented above.

The calculation of the number π by the Shanks method yields the following values: π ≌ 19/6, which is written in decimal form as: π ≌ 3.1(6), i.e., it has one correct decimal place and a relative error of δ = 0.8%.

The sum expression for the two-parameter approximation based on Newton's binomial is as follows:

$$\sum(S) \cong (1 + \text{M}\cdot x)^{\text{N}}, \text{ where } x = 1;\ \text{M} = 13/15,\ \text{N} = -5/13. \tag{15}$$

As a result, for the sum of the series, we get $\pi \cong 4\cdot(15/28)^{13/5} = 3{,}1463$. This result already has two correct decimal places and a relative error of $\delta = 0.14\%$, which is several times lower than in the Shanks method.

An attempt to calculate π within the new three-parameter approximation leads to complex-conjugate roots for the sum parameters ∑(S), since the determinant of the expression of the roots $a_{1,2}$ is negative: Det = $(B^2\text{-}A\cdot C)<0$.

Example 2.

Since the above number series converges slowly, we will use the number series converted from it by the same Ramaja with a faster convergence rate to calculate its sum (i.e., the number π):

$$S = 1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \ldots = 1 - \frac{(⅓)}{3} + \frac{(⅓)^2}{5} - \frac{(⅓)^3}{7} + \ldots, \tag{16}$$

$$\sum(S) = \frac{\pi}{\sqrt{12}}, \text{ where in the explicit form } \pi = (\sqrt{12}) \times \sum(S). \tag{17}$$

Let us transform this number series into the McLaurin series for $x = ⅓$, where

$$x = ⅓,\ a = -1/3;\ b = +1/5. \tag{18}$$

The calculation of the number π by the Shanks method gives the value for: $\pi \cong (7/3)^2/\sqrt{3}$, which is written in decimal form as: $\pi \cong 3.1433$, i.e., it has two correct decimal places and a relative error of $\delta = 1{,}8\cdot10^{-3}$.

The calculation of π by the MA method No. 1 gives the value: $\pi \cong (49/54)^{5/13}\cdot\sqrt{12}$, which is written in decimal form as: 3.1419, that is, it has three correct decimal places and a relative error of $\delta = 1{,}1\cdot10^{-4}$, which is an order of magnitude lower than the approximation error by the Shanks method.

Comparison of the results of calculating the number by Shanks' method (3.1433...) and the author's method (MA No. 1) based on Newton's binomial (3.1419...) showed that Shanks' method gives an accuracy of determining the number π slightly lower than in Archimedes' estimate (22/7 = 3.1422...), and the estimate by MA No. 1, on the contrary, gives an accuracy higher than 22/7 and even one more correct decimal place. Therefore, when analyzing the accuracy of the approximation, we will limit ourselves to the author's methods (two-parameter MA No. 1 and three-parameter MA No. 2).

The use of the three-parameter method to approximate the sum of this series leads to the fact that complex-conjugate numbers appear again in the expression of the approximate sum. It is interesting that even with complex roots of the parameters of the sum ∑(S) of the geometric type series (18) by the method of MA No. 2 - the sum of the Ramage number series for the value π is valid and can be easily calculated by MA No. 1 (3,141...).

Example 3.

Since the transformed series cannot be used to calculate the number π within the three-parameter approximation of its sum (Det<0), let's try to use another transformation of the number series (which is expected to give a series with Det>0) - find its inverse value: $S \to S^{-1}$. The idea is that if the coefficient A does not change in absolute value when transforming the power series S into the inverse series S^{-1} - then its sign A definitely changes to the opposite. Therefore, there is a possibility that the signs before the first (A) and third (C) coefficients will become different, and the sign of their product A·C in the expression Det=$(B^2\text{-}A\cdot C)$ will become negative. Then we can get Det>0 for S^{-1}.

From the expression $\sum(S) = \pi/\sqrt{12} = 1/\sum(S^{-1})$ it follows that: $\pi = (\sqrt{12})/\sum(S^{-1})$. Let's transform the Ramaji series S into the McLaurin series $S(x)$ for $x = 1$, and then the series $S(x) = 1 - x/9 + x^2/45 - x^3/189 + \ldots$ transform the inverse of it S^{-1} using the standard operation "series exponentiation" (see Dwight [9]). As a result of the transformation, we obtain the McLaurin series $S^{-1}(x)$:

$$S^{-1}(x) = 1 + \frac{1}{9}x - \frac{4}{405}x^2 - \frac{208}{25515}x^3 + \ldots, \text{ where } x = 1. \tag{19}$$

Now, in the case of Det>0, it is possible to approximate the sum of this series within the framework of the three-parameter approximation of MA No. 2.

The coefficients of the corresponding McLaurin series are as follows: A = +1/9; B = -4/405; C = -208/25515; (A·C)<0 and Det>0. We calculate the parameters of the sum of the base series ∑(S) using the formula: $a_{1,2} = (A/C)\cdot[B \pm \sqrt{(B^2 - A\cdot C)}\,]$. The substitution of the mianing gives: $a_{1,2} = (-0{,}1346)\cdot[-1 \pm \sqrt{(1+9{,}2857)}\,]$ ($a_1 = +0{,}5663$; $a_2 = -0{,}2971$), for the second parameter c we get $c_1 = +5{,}0971$; $c_2 = -2{,}6740$, for the third parameter we get: $b_1 = -0{,}8229$; $b_2 = +0{,}2265$.

The calculation of the sum of the inverse series for the first set of roots-parameters gives: $\pi = (\sqrt{12})/\sum(S^{-1}) \cong 3{,}1804$ (relative error $\delta = 1.2\%$), and for the second set of root-parameters we get: $\pi = (\sqrt{12})/\sum(S^{-1}) \cong 3{,}1559$, while the exact value of $\pi = 3.14159$ (relative error $\delta = 0.45\%$).

Thus, calculating the number π based on the inverse of the first Ramage series - using a three-parameter approximation of its sum - is possible in principle, but gives a higher error ($\delta = 0.45\%$) due to the additional transformation of the series. The new series S^{-1} has better convergence than S, but its first terms reflect the sum ∑(S) less accurately. Therefore, here the accuracy of determining the value of π by MA No. 2 was lower than the accuracy of its determination by MA No. 1.

4.3. Approximating the Sum of Number Series from Inverse Factorials

As a first example, let's use a rapidly converging number series that is not a familiar variable. It is used to determine the number *e*.

Example 1.

The base of natural logarithms, that is, the number $e = 1+\sum 1/n!$ [11]. And this series is formed by a sequence of inverse factorials of the numbers of the natural series, so although it is not familiar, it is fast convergent:

$$\sum\frac{1}{n!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots=1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\ldots \tag{20}$$

Let's transform this number series into the McLaurin series S(x) at x = 1:

$$\mathrm{S}(x)=1+\frac{1}{2}x+\frac{1}{6}x^2+\frac{1}{24}x^3+\ldots,\ \text{where } x=1. \tag{21}$$

The sum expression for the two-parameter approximation is as follows:

$$\sum(\mathrm{S})\cong(1+\mathrm{M}\cdot x)^{\mathrm{N}},\ \text{where } x=1,\ \text{parameters } a=+1/2,\ b=+1/6. \tag{22}$$

Find the parameters of the binomial: M = -1/6, and N = -3. As a result of the calculations, we get: $\mathrm{S}(1) \cong (5/6)^{-3}$. Or, in decimal form, we have $e \cong 1+(1{,}2)^3=2.728...$, while the exact result is e = 2.718. The relative error of the approximation by MA No. 1 is only δ = 0.7%.

And now, in the case of Det>0, we will approximate the sum of the same series, but within the framework of the three-parameter approximation by MA No. 2.

The coefficients of the corresponding McLaurin series are as follows: A = +1/2; B = +1/6; C = +1/24. $\mathrm{Det}=(\mathrm{B}^2-\mathrm{A}\cdot\mathrm{C})>0$. The parameters of the sum ∑(S) of the base series are calculated using the formula $a_{1,2} = (\mathrm{A/C})\cdot[\mathrm{B}\pm\sqrt{(B^2-A\cdot C)}\,]$. As a result: $a_{1,2} = 2\pm1$ ($a_1 = 3$; $a_2 = 1$), for the parameter c we get $c_{1,2} = 2\times(2\pm1) = 4\pm2$ ($c_1 = 6$; $c_2 = 2$), for the third parameter b we get: $b_1 = 3$; $b_2 = -1/3$.

The calculation of the refined sum of the series-base for the first set of roots-parameters gives $e = 1+\sum 1/n! \cong 1+43/25 = 1+1.72 = 2.72$, and for the second set of roots-parameters we get: $e = 1+\sum(1/n!) \cong 1+2/3 = 1+1{,}(6) = 2{,}(6)$, then the exact result is e = 2.718.

Thus, the first solution of the three-parameter approximation has a relative error of δ = 0.07%. That is, the accuracy of determining the number e by the method of MA No. 2 is higher than by MA No. 1 - by an order of magnitude!

Example 2.

As a second example, let's use a rapidly converging number series whose coefficients are positive. This number series is formed by a sequence of inverse factorials of the numbers of the natural series, and only odd ones, obtained by decomposing the series of the function sh(x):

$$\mathrm{sh}(1)=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots=1+\frac{1}{6}+\frac{1}{120}+\frac{1}{5040}+\ldots \tag{23}$$

Let's transform this numerical series into the McLaurin series at x = 1:

$$\mathrm{S}=1+\frac{1}{6}x+\frac{1}{120}x^2+\frac{1}{5040}x^3+\ldots,\ \text{where } x=1. \tag{24}$$

The sum expression for the two-parameter approximation is as follows:

$$\sum(\mathrm{S})\cong(1+\mathrm{M}\cdot x)^{\mathrm{N}},\ \text{where } x=1,\ \text{series coefficients } a=1/6,\ b=1/120. \tag{25}$$

We find the parameters of the binomial M = +1/15 and N = +5/2. As a result of the substitution, we obtain: $\mathrm{S}(1) \cong (16/15)^{5/2}$. Or in decimal form, $\sum(\mathrm{S}) \cong [1.0(6)]^{2.5} = 1.1751$. The exact result: sh(1) = 1.1752. The difference between the exact result and the estimate of the sum of the series by MA No. 1 is only $\sim10^{-4}$.

And now, in the case of Det>0, we will approximate the sum of the same series, but within the framework of the three-parameter approximation by MA No. 2.

The coefficients of the corresponding McLaurin series are as follows: A = +1/6; B = +1/120; C = +1/5040. $\mathrm{Det}=(\mathrm{B}^2-\mathrm{A}\cdot\mathrm{C})>0$. Calculate the parameters of the sum ∑(S) series using the formula: $a_{1,2} = (\mathrm{A\cdot B/C})\cdot[1\pm\sqrt{(1-\mathrm{A\cdot C/B^2})}\,] = 7\cdot[1\pm\sqrt{(11/21)}\,]$. As a result: $a_{1,2} = 7\pm5{,}0662$ ($a_1 = 12{,}0662$; $a_2 = 1{,}9338$), for the parameter c we get: $c_{1,2} = a_{1,2}/(1/6) = 6\times a_{1,2}$ ($c_1 = 72{,}397$; $c_2 = 11{,}603$ and for the third parameter $b_{1.2}$ we get: $b_1 = 31.609$; $b_2 = 0.3099$.

The calculation of the refined sum ∑(S) ≌ 1+0.1752 = 1.1752 of the basis series for the first set of roots-parameters gives: ∑(S) ≌ 1+0.1752 = 1.1752, while the exact result is sh(1) = 1.1752, i.e., we have four correct decimal places.

4.4. Attempting to Approximate the Sum of Divergent Number Series

Let's consider approximating the sum of divergent numerical series in order to determine the possibility of obtaining an answer to the question whether the series "as a whole" converges or not from a small initial fragment of the series.

Example 1.

As a first example, let's use probably the most famous divergent numerical series - the harmonic series, whose terms are the inverse of the numbers of the natural series.

The harmonic number series got its name because each of its terms is related to the previous and the next - by the "harmonic mean" ratio: $(a_n)^{-1} = (½)\cdot[(a_{n-1})^{-1}+(a_{n+1})^{-1}]$. This series has the following form:

$$S=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\ldots \tag{26}$$

Analysis of the sequence of terms in this series shows that they are all positive and monotonically decreasing. But it is not at all obvious that the harmonic series is divergent, although it has long been known (since 1673) that the sum of a harmonic number series ∑(S) = ∞. But we need to try to get an approximate value of its sum using the approximation methods discussed in this article in order to find out how to determine from a small fragment of the series whether it is converging or diverging.

This numerical series can be transformed into a McLaurin series at a fixed value of the variable x (x = 1):

$$S(x)=1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\ldots,\ \text{where } x = 1,\ a = 1/2,\ b = 1/3. \tag{27}$$

If we have a fragment of a series in which only the first two coefficients (+1/2 and +1/3) are known, then we can use the two-parameter method using the Newton binomial to approximate it:

$$\sum(S) \approx (1+M\cdot x)^N,\ \text{where } x = 1. \tag{28}$$

For this fragment of the series, the binomial parameters M = -5/6 and N = -3/5. When x = 1, the value of $\sum(S) \approx (1/6)^{-3/5}$ or in decimal form $\sum(S) \approx 6^{0.6} = 2.93$.

Thus, the two-parameter method of approximating the sum of the series MA No. 1 determines the value of ∑S(1) as the final value, i.e., having the values of the first two coefficients when estimating the sum of the series is not enough to establish that this series is divergent.

If the first three coefficients (+1/2, +1/3, and +1/4) are used to estimate the sum of the series, then we will try to use the three-parameter method to approximate the sum of the series. Analysis of the coefficients shows that the second and fourth coefficients have the same sign. Therefore, we need to calculate the determinant, which in this case is negative, since Det = $(1/3)^2$-(1/2)×(1/4) = 1/9-1/8 and 1/9<1/8. Thus, the roots of the quadratic equation for determining the parameters of the sum are complex-conjugate, and the expression for the sum of this series is complex. Therefore, we cannot use the three-parameter approximation here.

Now let's check how the inverse of the harmonic series converges - S^{-1}. Using the algorithm given in Dwight's Handbook [9], we will find the coefficients:

$$S^{-1}(x)=1-\frac{1}{3}x-\frac{4}{45}x^2-\frac{44}{945}x^3\ldots,\ x = 1;\ a = -1/2,\ b = -4/45. \tag{29}$$

If we have a fragment of the series in which only the first two coefficients (-1/3 and -4/45) are known, then we can use the two-parameter method of MA No. 2 using the Newton binomial to approximate it:

$$\sum(S^{-1}) \approx (1+M\cdot x)^N,\ \text{where } x = 1. \tag{30}$$

The parameters of this fragment are M = -5/6 and N = 3/5. For x = 1, the value $\sum(S^{-1})\approx 6^{-0.6}$ or in decimal form $\sum(S^{-1})\approx 0.341$. And the number $0.341 = (2.93)^{-1}$.

But if the first three coefficients (+1/3, +1/5, and +1/7) can be used to estimate the sum of the series, then we'll try to use the three-parameter method of MA No. 2 to approximate the sum of the series. Analysis of the coefficients shows that the second and fourth coefficients have the same sign. Therefore, it is necessary to calculate the determinant, which in this case is negative, since the determinant of Det = $(4/45)^2$ - (-1/3)×(-44/945) = (16/2025) - (44/2835) and (0.0079)<(0.0155). Thus, the roots of the quadratic equation for determining the parameters of the sum are complex-conjugate, and the expression for the sum of the series is also complex-conjugate. Therefore, in this case, we will not be able to use the three-parameter approximation to summarize the inverse series.

Example 2.

As a second example, let's use another divergent number series with ∑(S) = ∞, which is similar to the harmonic series, but only with those terms that contain odd numbers of the natural series:

$$S=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\cdots \tag{31}$$

This numerical series S can be transformed into the

McLaurin series S(x) at a fixed value of the variable x (x = 1)

$$\mathrm{S}(x)=1+\frac{1}{3}x+\frac{1}{5}x^2+\frac{1}{7}x^3 \ldots,\ \text{where } x = 1;\ a = 1/3,\ b = 1/5. \tag{32}$$

If there is a fragment of the series in which only the first two coefficients (+1/3 and +1/5) are known, then the two-parameter method using the Newton binomial can be used for approximation:

$$\sum(\mathrm{S}) \approx (1+\mathrm{M}\cdot x)^{\mathrm{N}},\ \text{where } x = 1. \tag{33}$$

For this fragment, M = -13/15 and N = -5/13. For x = 1, the value ∑(S) ≈ $(2/15)^{-5/13}$ or in decimal form ∑(S) ≈ $(15/2)^{5/13}$ = 2.17.

Thus, the two-parameter method of approximating the sum of the MA series No. 1 determines the value of S(1) as the final value, i.e., the values of the first two coefficients are not enough to say that this series is divergent.

If the first three coefficients (+1/3, +1/5, and +1/7) are used to estimate the sum of the series, then a three-parameter method can be used to approximate the sum of the series. Analysis of the coefficients shows that the second and fourth coefficients have the same sign. Therefore, it is necessary to calculate the determinant, which is negative in this case, since Det = $(1/5)^2$-(1/3)×(1/7) = 1/25-1/21, but 1/25<1/21. Thus, the roots of the quadratic equation for determining the parameters of the sum of a series are complex-conjugate, and the expression for its sum is also complex-conjugate. Therefore, the three-parameter approximation cannot be used here either.

Now let's check whether the inverse of S^{-1} converges to the series discussed above. Using the algorithm given in Dwight's Handbook [9], we calculate its coefficients:

$$\mathrm{S}^{-1}(x)=1-\frac{1}{3}x-\frac{4}{45}x^2-\frac{44}{945}x^3 \ldots,\ x = 1;\ a = -1/2,\ b = -4/45. \tag{34}$$

If we have a fragment of the series in which only the first two coefficients are known (-1/3 and -4/45), then we can use the two-parameter method No. 2 using the Newton binomial to approximate it:

$$\sum(\mathrm{S}^{-1}) \approx (1+\mathrm{M}\cdot x)^{\mathrm{N}},\ \text{where } x = 1. \tag{35}$$

For this fragment, M = -13/15, and N = +5/13. The value of S(1) ≈ $(2/15)^{5/13}$ or in decimal form ∑(S^{-1}) ≈ 0.46. And the number 0.46 = $(2.17)^{-1}$.

But if you use the first three coefficients (+1/3, +1/5 and +1/7) to estimate the sum of the series, you can use the three-parameter method to approximate the sum of the series. Analysis of the coefficients shows that the second and fourth coefficients have the same sign. Therefore, we need to calculate the determinant, which in this case is negative: Det = $(4/45)^2$ - (-1/3)×(-44/945) = (16/2025) - (44/2835) and (0.0079)<(0.0155). Thus, the roots of the quadratic equation for determining the parameters of the sum are complex-conjugate, and the expression for the sum of the series is also complex-conjugate. Therefore, we cannot use the three-parameter approximation to summarize the inverse series either.

Conclusion. Both methods (MA No. 1 and No. 2) have quite acceptable accuracy for estimating the sum of series, and of completely different nature - both numerical and power series, in particular, of geometric type. But if MA No. 1 in all cases has real numbers as sum parameters, then MA No. 2 for rapidly converging series {for them $(a_n)^2 \geq (a_{n-1}\times a_{n+1}$ gives real values of the sum, and for slowly converging and diverging series {for them $(a_n)^2 < (a_{n-1}\times a_{n+1})$}, it gives complex-conjugate roots of their sum parameters.

Using a two-parameter approximation (MA No. 1), it is impossible to determine from a fragment of a power series that it is diverging, since it gives a finite number as a result of summation, both for converging and diverging series (for diverging series, it is incorrect). After all, the exact result of sum is ∑(S) = ∞, and ∑(S^{-1}) = 0.

4.5. Three-Parameter Approximation of the Sum of Lord Rayleigh's Power Series

More than a hundred years ago, the main method of measuring the surface tension of a liquid was the method of raising the liquid in a capillary - a thin glass tube of constant radius, provided that $r \ll h$, where h is the height of the liquid in a vertical capillary. In this case, the experiment measures not the distance H from "zero" to the edge of the meniscus (it is difficult to fix it), but the distance from the liquid level in the vessel to the bottom of the meniscus in the capillary. However, the meniscus, which "hangs" on the capillary walls due to the wetting forces, forms an edge angle θ with them, so it increases the weight of the liquid relative to what would occur when measuring h along the edge. This makes the problem of accurately determining the "height of liquid rise" in a capillary uncertain [12].

In 1916, Lord Rayleigh [13] analytically (using the small parameter method) solved the problem of determining the surface tension of a liquid from the experimental data during capillary rise and obtained a formula for the dependence of the capillary complex [14] of a liquid a^2 ($a^2 = 2\sigma/\Delta\rho g$) in the form: $a^2=(r\cdot h)\times\sum\mathrm{S}(r/h)$, where S($r/h$) is a fragment of the series - i.e., the expansion by the small parameter ε, where ε is the ratio (r/h) << 1. Lord Rayleigh obtained the first four terms of the power series S, in which the dependence a^2 = ($r\cdot h$)×f(r, h) is expanded:

$$S\left(\frac{r}{h}\right) = 1 + \frac{1}{3}\varepsilon - 0{,}1288\varepsilon^2 + 0{,}1312\varepsilon^3 - \ldots, \text{ where } \varepsilon = r/h. \tag{36}$$

However, this analytical solution was practically not widespread because it includes a fragment of the power series, the accuracy of which (in contrast clozed formula [15]) was questionable (uncertain) in calculating the sum of the resulting expansion $\sum S(r/h)$.

Therefore, we will first solve the problem of approximating the power series (44) by the MA No. 1, using only the first three terms of the expansion:

$$S\left(\frac{r}{h}\right) = 1 + \frac{1}{3}\varepsilon - 0{,}1288\varepsilon^2 + \ldots, \text{ where } \varepsilon = r/h. \tag{37}$$

We use Newton's binomial for this purpose: $\sum S(x) = (1 + M\cdot x)^N$, where $M = a\cdot(1-2b/a^2)$, and $N = (1-2b/a^2)^{-1}$, the series coefficients are $a = +1/3$, $b = -0.1288$. Substitution gives M = 1.0106. The parameter N = 0.3013. As a result, for $\sum S(r/h)$ we obtain:

$$a^2 = (r\cdot h) \times \sum S(\varepsilon), \text{ where } \sum S(\varepsilon) \cong (1 + 1{,}0106\cdot\varepsilon)^{0{,}3013}. \tag{38}$$

The above formula (46) can be used as a first approximation. The curve of this dependence $y = f(\varepsilon)$ can be represented in double logarithmic coordinates - the results of meniscus elevation in capillaries of different radius r should "lie" on a straight line and can be processed statistically, for example, by the least squares method:

$$ln\left(\frac{a^2}{r\cdot h}\right) \sim 0{,}3013\cdot ln\left[1 + 1{,}0106\cdot\left(\frac{r}{h}\right)\right], \tag{39}$$

where on the ordinate axis we plot the value of $y(x) = ln[a^2/(r\cdot h)]$ and on the abscissa axis - the value of $x = ln(1 + 1{,}0106\cdot\varepsilon)$.

Let us approximate the sum of the Rayleigh series by the three-parameter method of MA No. 2. First, we calculate the determinant by the coefficients of the series:

$$S(\varepsilon) = 1 + \frac{1}{3}\varepsilon - 0{,}1288\varepsilon^2 + 0{,}1312\varepsilon^3 - \ldots, \text{ for this fragment Det<0.} \tag{40}$$

Therefore, the primary series must be transformed: we will square it using procedure No. 51.1 from reference [7]. We get the expression for the square of the series $S^2(\varepsilon)$:

$$S^2(\varepsilon) = 1 + \frac{2}{3}\varepsilon + 0{,}\ 3667\varepsilon^2 + 0{,}1765\varepsilon^3 + \cdots \tag{41}$$

The calculation of the determinant gives: $B^2 = 0.1345$, $A\cdot C = 0.1177$; $Det = B^2 - A\cdot C = +0.01678$, i. e. Det > 0 and the root $\sqrt{(B^2 - A\cdot C)} = 0.1296$.

For the first parameter a, we get: $a_{1.2} = 3.7765\cdot(2/3\pm0.1296)$. As a result, for the roots we have: $a_1 = 3.007$, $a_2 = 2.0284$; $c_1 = 4.5104$, $c_2 = 3.0426$; $b_1 = 4.7156$, $b_2 = 1.3848$.

Substituting the first set of roots ($a_1 = 3.007$, $c_1 = 4.5104$, $b_1 = 4.7156$) into the expression for approximating the sum of the series $\sum S^2(r/h)$ by MA No. 2, we find:

$$\sum S^2(r/h) \cong 1 + \varepsilon\cdot\frac{1{,}5622 + 1{,}7087\varepsilon}{(4{,}5104 - \varepsilon)^2}, \text{ where } \varepsilon = r/h. \tag{42}$$

Substituting the second set of roots ($a_2=2.0284$, $c_2=3.0426$, $b_2=1.3848$) into the expression for the approximation of the sum of the series $\sum S^2(r/h)$ by MA No. 2, we obtain:

$$\sum S^2(r/h) \cong 1 + \varepsilon\cdot\frac{6{,}1717 - 0{,}6436\varepsilon}{(3{,}0426 - \varepsilon)^2}, \text{ where } \varepsilon = r/h. \tag{43}$$

Comparison of the two solutions shows that the domain of $\varepsilon = r/h$ in the first case is $0 < \varepsilon < 4.5104$, while in the second case it is much narrower: $0 < \varepsilon < 3.0426$ - almost one and a half times. Therefore, the choice of the solution is obvious - the second solution is rejected, and the first one is taken as a basis. The analysis of the first solution showed that its accuracy in the range $r/h<1$ will be quite high, since it is much narrower than the range of r/h determination.

The calculation of the capillary complex should be carried out taking into account the fact that we did not use the series S, but its square S^2, so in the expression of the capillary complex, we need to find the square root of the sum of the series:

$$a^2=(r\cdot h)\times\sqrt{\left\{\sum \mathrm{S}^2(\varepsilon)\right\}}=(r\cdot h)\times\sqrt{\left\{1+\varepsilon\cdot\frac{1,5622+1,7087\cdot\varepsilon}{(4,5104-\varepsilon)^2}\right\}}, \quad (44)$$

where $\varepsilon=r/h$, hence the surface tension: $\sigma=a^2\Delta\rho g/2$ (where $a^2 = 2\sigma/\Delta\rho g$).

5. Conclusions

As the analysis of the three-parameter approximation of the sum of the McLaurin series (MA No. 2) developed in this article - based on a geometric-type series-base - has shown, its accuracy is significantly (by an order of magnitude) higher than the accuracy of the author's two-parameter approximation (MA No. 1) based on the Newton binomial, although MA No. 1 has a very high accuracy - when evaluating the number π by the sum of the series, it gives three correct decimal places, and when evaluating the number *e*, the error is 0.7%.

Although the approximation accuracy of MA No. 2 is a few decimal places higher than that of MA No. 1, it may decrease if the primary series must be transformed into a series that converges faster to meet the condition $a^2_n\leq(a_{n-1}\times a_{n+1})$ And in this case, the accuracy of approximating the sum of the series decreases. This is obviously due to the fact that MA No. 2 is based on the generalization of a geometric series with an exact sum expression, in which the neighboring terms are related by the following relationship: $a^2_n=(a_{n-1}\times a_{n+1})$, unlike MA No. 1, where there is no such strict condition.

The three-parameter approximation is unique in that it provides a new quality - it does not allow "summing" divergent series, since for both the sum of the primary series $\sum$S and the sum of the inverse series $\sum S^{-1}$ it produces complex-conjugate roots in the expression of the parameters relative to the expression of the approximation of the sum of the base series.

It can be said that the use of a three-parameter approximation of the sum of a series when interpreting fragments of power series obtained by physicists when solving nonlinear problems by the small parameter method will allow obtaining not only quantitatively more accurate results based on them: $R(x^4)$, but also their qualitatively new physical interpretation.

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